5.2

Downtown and Uptown

Graphs of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Solve exponential functions using the intersection of graphs.
- Analyze asymptotes of exponential functions and their meanings in context.
- Identify the domain and range of exponential functions.
- Analyze and graph decreasing exponential functions.
- Compare graphs of linear and exponential functions through intercepts, asymptotes, and end behavior.

KEY TERM

- horizontal asymptote

As the world economy has shifted from agriculture to industry over the past century, more and more people live in cities. This trend is known as urbanization.

People tend to move to cities for career opportunities and more cultural interactions. In 1900, only 13% of the world's population lived in urban environments. By 2005, this figure rose to 49%. Some experts have predicted that by 2030, 60% of the global population will consist of city dwellers.

One negative effect of urbanization can be to drive prices dramatically higher. In Manila, one of the densest cities in the world, skyrocketing prices have forced many people to live in slums.

PROBLEM 1 Downtown and Uptown

At this moment, the population of Downtown is 20,000, and the population of Uptown is 6000. But over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. What are the independent and dependent quantities in each situation?

2. Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?

Let's analyze the population growth of Uptown. In 1 year from now, the population of Uptown will be

$$6000 + 6000(0.018) = 6108.$$

The population will be 6108 people in Uptown 1 year from now.

3. Write and simplify an expression that represents the population of Uptown:
   - **a.** 2 years from now.
   - **b.** 3 years from now.

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4. How can you tell that this function is an exponential function? Explain your reasoning.

You can use the formula for compound interest to determine the function for Uptown's increasing population. Recall that the formula for compound interest is $P(t) = P(1 + r)^t$, where $P(t)$ represents the amount in the account after a certain amount of time in years, r is the interest rate written as a decimal, and t is the time in years.

5. In the compound interest formula, substitute Uptown's starting population for P and the rate of population growth for r.
   - **a.** Write the function, $U(t)$, showing Uptown's population growth as a function of time in years.
   - **b.** Use your answers to Question 3 and a calculator to verify that your function is correct.

Now let's analyze the population decline of Downtown.

6. Write and simplify an expression that represents the population of Downtown. The first one has been done for you.

a. 1 year from now.

$20{,}000 - 20{,}000(0.015) = 19{,}700$

The population of Downtown will be 19,700 people 1 year from now.

b. 2 years from now.

c. 3 years from now.

7. Rewrite the expressions for the population decline in Downtown using the Distributive Property. The first one has been done for you.

a. 1 year from now.

$20{,}000 - 20{,}000(0.015)$

$20{,}000(1 - 0.015)$

b. 2 years from now.

c. 3 years from now.

8. Use the compound interest formula and your expressions in Question 7 to write the function, $D(t)$, showing Downtown's population decline as a function of time in years.

9. Think about each function as representing a sequence. What is the common ratio in simplest form, or the number that is multiplied each time to get the next term, in each sequence?

10. Explain how the common ratios determine whether the exponential functions for the change in population are increasing or decreasing.

PROBLEM 2 Graphing, Finally!

Let's examine the properties of the graphs of the functions for Downtown and Uptown. Here are the functions again:

Downtown: $D(t) = 20{,}000(1 - 0.015)^t$ Uptown: $U(t) = 6000(1 + 0.018)^t$

1. Use a graphing calculator to graph both functions using the bounds $[-100, 100] \times [0, 30{,}000]$.

2. Let's analyze the y-intercepts of each function.
   a. Identify the y-intercepts.

   b. Interpret the meaning of the y-intercept in terms of this problem situation.

   c. Describe how you can determine the y-intercept of each function using just the formula for population increase or decrease.

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3. Use a graphing calculator to answer each question. Describe your strategy.
   a. How long will it take for Downtown's population to be half of what it is now?

   b. How long will it take for Uptown's population to double from what it is now?

c. How many years from now will the populations of Downtown and Uptown be equal? Determine the approximate populations.

4. Use the **TRACE** function on your calculator to determine when the population of Downtown was 0. Then, determine when the population of Uptown was 0. What do you notice?

5. Determine the point at which each function is equal to 0 by graphing the line $y = 0$ and determining the intersection point of this line with each function. What do you notice?

Each population function you graphed has a *horizontal asymptote*. A **horizontal asymptote** is a horizontal line that a function gets closer and closer to, but never intersects.

6. Write the equation for the horizontal asymptote of each population function.

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7. Does the horizontal asymptote make sense in terms of this problem situation? Explain your reasoning.

8. Identify the domain and range of each function.

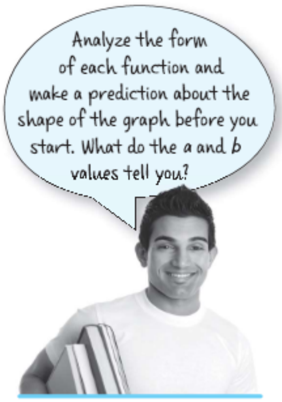
PROBLEM 3 The Multiple Representations of Exponentials

1. Complete the table and sketch a graph for each exponential function of the form $f(x) = ab^x$. Then determine the x-intercept(s), y-intercept, asymptote, domain, range, and interval(s) of increase/decrease.

a. $f(x) = 3^x$

x	$f(x)$
−2	
−1	
0	
1	
2	
3	

x-intercept(s):
y-intercept:
asymptote:
domain:
range:
interval(s) of increase/decrease:



b. $g(x) = \left(\frac{1}{2}\right)^x$

x	$g(x)$
−2	
−1	
0	
1	
2	
3	

x-intercept(s):
y-intercept:
asymptote:
domain:
range:
interval(s) of increase/decrease:

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c. $k(x) = 5 \cdot 2^x$

x	$k(x)$
−2	
−1	
0	
1	
2	
3	

x-intercept(s):

y-intercept:

asymptote:

domain:

range:

interval(s) of increase/decrease:

d. $p(x) = -4^x$

x	$p(x)$
−2	
−1	
0	
1	
2	
3	

x-intercept(s):

y-intercept:

asymptote:

domain:

range:

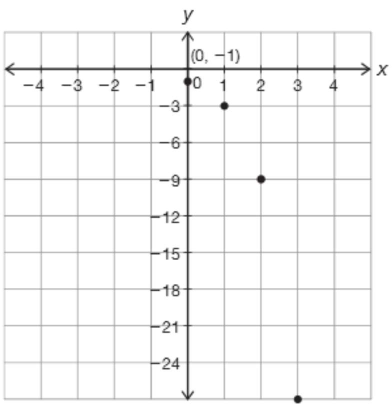
interval(s) of increase/decrease:

2. Write an exponential equation of the form $y = ab^x$ for each. Explain your reasoning.

a.

x	y
0	1
1	4
2	16
3	64

b.



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3. Given a function of the form $f(x) = ab^x$.

a. What does the a-value tell you?

b. What does the b-value tell you?

Be prepared to share your solutions and methods.